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ON LOGARITHMIC DERIVATIVES OF THE ASSOCIATED LEGENDRE FUNCTIONS OF ARBITRARY COMPLEX DEGREE

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In solving certain problems of the theory of vibrations of spherical shells it is more convenient to calculate not the associated Legendre functions $P_n^m(\cos\theta)$ and their derivatives themselves but rather the logarithmic derivatives

$$F_n^m(\cos\theta) = \frac{d}{d\theta}[\ln P_n^m(\cos\theta)] = \frac{d}{d\theta} P_n^m(\cos\theta) \,/\, P_n^m(\cos\theta)$$

We consider the case $\theta = \pi/2$, when it is possible to calculate the logarithmic derivative of $P_n^m(\cos\theta)$, where $n = u + i\tau$ is an arbitrary complex number, without the use of hypergeometric series. Using the well known expressions for the function $P_n^m(0)$ and its first derivative in terms of the gamma function [1], we obtain

$$F_n^m(0) = -2\frac{\Gamma(1+l_+)\Gamma(1+l_-)}{\Gamma(^1/_2+l_+)\Gamma(^1/_2+l_-)}\operatorname{tg}(l_+\pi), \quad l_\pm = \frac{n\pm m}{2} \tag{1}$$

In what follows we shall need to distinguish the cases corresponding to odd or even values for the order m of the function $P_n^m(0)$. We shall make repeated application of the recursion formula $\Gamma(z+1) = z\Gamma(z)$ to each of the gamma functions appearing in the expression (1); we also take into account the relation [2]

$$(1-n)\left(1+\frac{n}{2}\right)\left(1-\frac{n}{3}\right)\left(1+\frac{n}{4}\right)\ldots = \sqrt{\pi}\left[\Gamma\left(1+\frac{n}{2}\right)\Gamma\left(\frac{1}{2}-\frac{n}{2}\right)\right]^{-1}$$

After a number of operations are carried out the resulting expressions for the logarithmic derivatives of $P_n^m(0)$ are found to be

$$F_n^m(0) = \prod_{s=1,3,5,\ldots}^{m} A_s \prod_{k=1,3,5,\ldots}^{\infty} B_k \Big/ \prod_{k=2,4,6,\ldots}^{\infty} B_k \qquad (\text{odd } m) \tag{2}$$

$$F_n^m(0) = -p \prod_{s=2,4,6,\ldots}^{m} A_s \prod_{k=2,4,6,\ldots}^{\infty} B_k \Big/ \prod_{k=1,3,5,\ldots}^{\infty} B_k \qquad (\text{even } m)$$

$$A_s = \frac{p - s(s-1)}{p-(s-1)(s-2)}, \quad B_k = 1 - \frac{p}{k(k+1)}, \quad p = n(n+1)$$

Keeping the degree n the same but letting the order m vary, we can calculate the functions $F_n^m(0)$ from the recursion formulas

$$F_n^{m+1}(0) = [m(m+1) - p] \,/\, F_n^m(0), \qquad F_n^{m+2}(0) = A_{m+2} F_n^m(0)$$

To derive an asymptotic expression for $F_n^m(\cos\theta)$ for large values of τ and arbitrary angle θ we use a trigonometric expansion of the associated Legendre functions [1]. Assuming the quantity τ to be so large that $\operatorname{sh}\tau\theta \approx \operatorname{ch}\tau\theta \approx e^{\tau\theta}/2$, we obtain the asymptotic formulas (α, β and φ_0 are real)

$$P_n^m(\cos\theta) \approx \frac{\exp(\tau\theta + \alpha)}{\sqrt{2\pi\sin\theta}}\left[\cos(\varphi_0 - \beta) - i\sin(\varphi_0 - \beta)\right] \tag{3}$$

$$\alpha + i\beta = \ln\left[\Gamma(n + m + 1) / \Gamma(n + {}^3/_2)\right]$$

$$\varphi_0 = \left(u + \frac{1}{2}\right)\theta + \left(m - \frac{1}{2}\right)\frac{\pi}{2}, \quad u = \operatorname{Re} n$$

From the formulas (3) it follows that

$$F_n^m(\cos\theta) \approx \tau - {}^1/_2 \operatorname{ctg}\theta - i(u + {}^1/_2)$$

Thus for large τ the logarithmic derivatives of the associated Legendre functions are practically independent of the order m.

REFERENCES

1. Hobson, E. W., The Theory of Spherical and Ellipsoidal Harmonics. Chelsea, New York, 1955.
2. Bateman, H. and Erdelyi, A., Higher Transcendental Functions. Vols. 1 and 2, McGraw-Hill, New York, 1953.

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